SQUARE ROOT PROBLEM OF KATO FOR THE SUM OF OPERATORS

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Abstract. This paper is concerned with the square root problem of Kato for the "sum" of linear operators in a Hilbert space $\mathbb{H}$. Under suitable assumptions, we show that if A and B are respectively m-scetroial linear operators satisfying the square root problem of Kato. Then the same conclusion still holds for their "sum". As application, we consider perturbed Schrödinger operators.

1. Introduction

In this paper we deal with the square root problem of Kato for the sum of linear operators in a Hilbert space $\mathbb{H}$. Indeed, let A, B be (unbounded) m-sectorial operators in a (complex) Hilbert space $\mathbb{H}$ and let Φ and Ψ be the (sectorial) sesquilinear forms associated with A and B respectively by the first representation theorem, see, e.g., [16, Theorem 2.1, p. 322]. We say that A and B verify *the square root problem of Kato* if the following holds

$$D(A^{\frac{1}{2}}) = D(\Phi) = D(A^{*\frac{1}{2}}) \quad \text{and} \quad D(B^{\frac{1}{2}}) = D(\Psi) = D(B^{*\frac{1}{2}}) \tag{1}$$

Our primary goal in this paper is to prove that if (1) holds and under suitable assumptions, then the same conclusion still holds for the algebraic sum $A + B$, that is,

$$D((A+B)^{\frac{1}{2}}) = D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}}) = D((A+B)^{*\frac{1}{2}}) \tag{2}$$

As consequence, we shall discuss the particular case of unbounded normal operators defined in a (complex) Hilbert space $\mathbb{H}$.

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[2]Key words: square root problem of Kato, sesquilinear forms, m-sectorial operators, algebraic sum, sum form.

It is well-known that the algebraic sum $A+B$ of A and B is not always defined (see [7], [8], and [9]). To overcome such a difficulty, we shall also deal with an extension of the algebraic sum called *sum form*. Recall that more details about the sum form $A \dot{+} B$ of A and B, can be found in [16, 5. Supplementary remarks, p. 328-32] or [6]. One then can show that if (1) holds, and under appropriated assumptions; then the same conclusion still holds for the sum form, that is,

$$D((A \dot{+} B)^{\frac{1}{2}}) = D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}}) = D((A \dot{+} B)^{*\frac{1}{2}}) \tag{3}$$

In [22], McIntosh has shown that if C is an invertible m-accretive operator in a Hilbert space $\mathbb{H}$ such that its spectrum $\sigma(C)$ is a subset of a region of type $S_{\alpha,\beta} = \{z \in \mathbb{C} : \Re ez \geq 0 \text{ and } |\Im mz| \leq \beta(\Re ez)^{\alpha}\}$, where $\alpha \in [0,1)$ and $\beta > 0$. Then $D(C^{\frac{1}{2}}) = D(C^{*\frac{1}{2}})$. In section 3, a similar result will be discussed for the sum of invertible m-accretive operators.

Historically, the well-known square problem of Kato takes its origin in a remark formulated in [16, Remark 2.29, p. 332-333]. It drew the attention of several mathematicians, especially the pioneer work of McIntosh.

Recall that the first counter-example to the square root problem in the general case of abstract m-accretive operators, formulated by Kato, was found by Lions in [19], that is,

$$D(C) = \mathbb{H}^1_0(0,+\infty) \quad \text{and} \quad Cu = \frac{d}{dt}u, \quad \forall u \in D(C) \tag{4}$$

Clearly, C is m-accretive (not m-sectorial) and that: $D(C^{\frac{1}{2}}) \neq D(C^{*\frac{1}{2}})$.

A few years later, a remarkable counter-example to the square root problem for the general class of abstract m-sectorial operators was found by McIntosh. Indeed, in [21], it is shown that there exists an m-sectorial operator A such that $D(A^{\frac{1}{2}}) \neq D(A^{*\frac{1}{2}})$. Meanwhile, McIntosh and allies kept investigating on the square root problem of Kato for elliptic linear operators, formulated by Kato in [14]. Such a question was modified by McIntosh in [22]. Recently, such a famous and challenging question has been solved by McIntosh and allies. Indeed, they have proven that the domain of the square root of a uniformly complex elliptic operator $A = -div(B\nabla)$ with bounded measurable coefficients in $\mathbb{R}^n$ is the Sobolev space $\mathbb{H}^1(\mathbb{R}^n)$ with

the estimate: $\|A^{\frac{1}{2}}u\|_{L^2} \sim \|\nabla u\|_{L^2}$, where $\sim$ is the equivalence in the sense of norms, see, e.g., [2] and [3] for details.

2. Preliminaries

2.1. **Notation and Definitions.** Throughout the paper, $\mathbb{R}$, $\mathbb{C}$, $(\mathbb{H}, \langle,\rangle)$, $B(\mathbb{H})$ stand for the sets of real, complex numbers, a (complex) Hilbert space endowed with the inner product $\langle,\rangle$ and the space of bounded linear operators, respectively; $S_{\alpha,\beta}$ denotes the domain of the complex plan defined by: $S_{\alpha,\beta} = \{z \in \mathbb{C} :\ \Re ez \geq 0 \text{ and } |\Im mz| \leq \beta(\Re ez)^{\alpha}\}$, where $\alpha \in [0, 1)$ and $\beta > 0$.

For a linear operator A, we denote by $D(A)$, $\sigma(A)$ the domain and the spectrum of A. For a given sesquilinear form $\Phi :\ D(\phi)\times D(\phi) \subset \mathbb{H}\times\mathbb{H} \mapsto \mathbb{C}$, we denote by $\Theta(\phi)$, its numerical range defined by: $\Theta(\phi) = \{\phi(u, u) :\ u \in D(\phi) \ \text{ with } \ \|u\| = 1\}$. Similarly, the numerical range of a given linear operator A is defined by: $\Theta(A) = \{\langle Au, u\rangle :\ u \in D(A) \ \text{ with } \ \|u\| = 1\}$.

Below we list some properties of sectorial sesquilinear forms as well as m-sectorial operators that we shall use in the sequel.

Definition 2.1. A sesquilinear form $\Phi :\ D(\phi) \times D(\phi) \mapsto \mathbb{C}$ is said to be sectorial if $\Theta(\Phi)$ is a subset of the sector of the form

$$\mathcal{S}_{\alpha,\beta} = \{\lambda \in \mathbb{C} :\ |\arg(\lambda - \beta) \leq \alpha < \frac{\pi}{2}\},$$

where $\beta \in \mathbb{R}$.

Remark 2.2. Throughout this paper, we assume that $\beta = 0$. In this case

$$|\Im m\Phi(u, u)| \leq \tan\alpha\ \Re e\Phi(u, u), \quad \forall u \in D(\Phi), \tag{5}$$

where $\Re e\ \Phi = \dfrac{1}{2}(\Phi + \Phi^*)$ and $\Im m\ \Phi = \dfrac{1}{2}(\Phi - \Phi^*)$ with Φ^* denotes the conjugate of the sesquilinear Φ (see [16]).

Definition 2.3. A linear operator $A :\ D(A) \subset \mathbb{H} \mapsto \mathbb{H}$ defined on $\mathbb{H}$ is said to be m-accretive if the following statements hold true

(i) $\Re e \, \langle Au, u \rangle \; \geq \; 0$

(ii) $(A + \lambda \, I)^{-1} \in B(\mathbb{H})$ and $\|(A + I\lambda)^{-1}\| \leq \dfrac{1}{\Re e \; \lambda}, \quad \Re e \; \lambda > 0$

Example 2.4. Let Ω be a bounded open subset of $\mathbb{R}^n$ and let A be the operator defined by

$$D(A) = \mathbb{H}_0^1(\Omega) \cap \mathbb{H}^2(\Omega) \;\text{ with }\; Au = -\Delta u,$$

where $\Delta = \sum_{k=1}^n \dfrac{\partial^2}{\partial x_k^2}$ denotes the Laplace differential operator. Clearly, A is (self-adjoint) m-accretive in the Hilbert space $L^2(\Omega)$.

Definition 2.5. A linear operator $A: \; D(A) \subset \mathbb{H} \mapsto \mathbb{H}$ defined on $\mathbb{H}$ is said to be quasi-m-accretive if $A + \xi I$ is m-accretive for some scalar ξ.

Definition 2.6. A linear operator $A: \; D(A) \subset \mathbb{H} \mapsto \mathbb{H}$ defined on $\mathbb{H}$ is said to be sectorial if $\Theta(A) \subseteq \mathcal{S}_{\alpha \, , \, \beta}$. The operator A is said to be m-sectorial if A is sectorial and quasi-m-accretive.

Let Φ be a sectorial form in the Hilbert space $\mathbb{H}$. We denote by $\mathbb{H}_\Phi$, the Pre-Hilbert space $D(\Phi)$, when equipped with the inner product given by

$$\langle u, v \rangle_\Phi \; = \; \Re e \Phi(u, v) + \langle u, v \rangle, \quad \forall u, v \in D(\Phi) \tag{6}$$

It can be shown that $\mathbb{H}_\Phi$ is a Hilbert space if and only if Φ is a densely defined closed sectorial form.

We also need the following theorem due to Lions (see [19]).

Theorem 2.7. *Let A be an m-sectorial operator on $\mathbb{H}$ and let Φ be the densely defined closed sectorial form associated with A. Assume that there exists a Hilbert space $\mathbb{K} \hookrightarrow \mathbb{H}$ such that*

(i) *$D(\Phi)$ is a closed subspace of $[\mathbb{K}, \mathbb{H}]_{\frac{1}{2}}$*

(ii) *$D(A) \subset \mathbb{K}$ and $D(A^*) \subset \mathbb{H}$*

Then

$$D(A^{\frac{1}{2}}) = D(\Phi) = D(A^{*\frac{1}{2}})$$

Below we list some properties of the "sum" of operators which we will need in the sequel.

2.2. **Sum of Operators.** Let A, B be m-sectorial operators on $\mathbb{H}$. Their algebraic sum is defined by

$$D(A+B) = D(A) \cap D(B), \quad (A+B)u = Au + Bu \quad \forall u \in D(A) \cap D(B)$$

It is well-known that the algebraic sum defined above is not, always defined. A typical example can be formulated by the following: Set $\mathbb{H} = L^2(\mathbb{R}^3)$ and consider A, B, be the m-sectorial operators given by

$$D(A) = \mathbb{H}^2(\mathbb{R}^3), \quad Au = -\Delta u, \ \ \forall u \in \mathbb{H}^2(\mathbb{R}^3)$$

and

$$D(B) = \{u \in L^2(\mathbb{R}^3) : \ \ V(x)u \in L^2(\mathbb{R}^3)\}, \quad Bu = Vu, \ \ \forall u \in D(B)$$

where V is a complex-valued function satisfying the following assumption

$$\Re e\, V > 0, \quad V \in L^1(\mathbb{R}^3) \ \text{ and } \ V \notin L^2_{loc}(\mathbb{R}^3) \tag{7}$$

Proposition 2.8. Let A, B be the linear operators given above. Assume that the assumption (7) holds. Then $D(A) \cap D(B) = \{0\}$.

Proof. Let $u \in D(B) \cap D(B)$ and assume that $u \not\equiv 0$. Since $u \in \mathbb{H}^2(\mathbb{R}^3)$; then u is a continuous function according to the theorem of Sobolev (see [1]). Thus, there are an open subset Ω of $\mathbb{R}^3$ and $\delta > 0$ such that $|u(x)| > \delta$ for all $x \in \Omega$. Let Ω' be a compact subset of Ω, equipped with the induced topology by Ω (Ω' is also a compact subset of $\mathbb{R}^3$). It follows that,

$$|V|_{\Omega'} = \frac{(|Vu|)_{\Omega'}}{|u|_{\Omega'}} \in L^2(\Omega'), \tag{8}$$

Indeed, $(|Vu|)_{\Omega'} \in L^2(\Omega')$ and $\dfrac{1}{(|u|)_{\Omega'}} \in L^\infty(\Omega')$. Thus, $V \in L^2(\Omega')$; this is impossible according to the assumption (7)($V \notin L^2_{loc}(\mathbb{R}^3)$). Therefore $u \equiv 0$. ∎

As the previous example shows, the domain of the algebraic sum $A+B$ of A and B must be watched carefully. To overcome such a difficulty, we define an extension of the algebraic sum commonly called sum form, defined with the help of the sum of sesquilinear forms. Indeed, let A, B be m-sectorial operators on $\mathbb{H}$ and let Φ and Ψ be the sesquilinear forms associated with A and B respectively. It is well-known that Φ and Ψ are respectively densely defined closed sectorial sesquilinear forms. In addition, we have

$$\Phi(u,v) = \langle Au, v\rangle, \quad \text{for every } \ u \in D(A) \ \text{ and } \ v \in D(\Phi) \supset D(A)$$

and

$$\Psi(u,v) = \langle Bu, v\rangle, \quad \text{for every } \ u \in D(B) \ \text{ and } \ v \in D(\Psi) \supset D(B)$$

Now consider their sum defined by,

$$D(\Xi) = D(\Phi) \cap D(\Psi) \quad \text{and} \quad \Xi = \Phi + \Psi$$

Assume that $\overline{D(\Phi) \cap D(\Psi)} = \mathbb{H}$; then Ξ is a densely defined closed sectorial sesquilinear form (see [16, Theorem 1.31, p. 319]). Using the first representation theorem to the sectorial sesquilinear form Ξ (see [16, Theorem 2.1, p.322]); it turns out that there exists a unique m-sectorial operator associated with it; we denote it by $A \dot{+} B$ and call it as the sum form of A and B.

Let us notice that the sum form $A \dot{+} B$ defined in this way is the m-accretive extension of the closure $\overline{A+B}$ (if defined) of $A+B$. Furthermore, $A \dot{+} B$ and $\overline{A+B}$ coincide if this last is a maximal accretive operator. Therefore, the sum form $A \dot{+} B$ is defined even if $A+B$ is not.

3. Main Results

Theorem 3.1. *Let A and B be m-sectorial linear operators on $\mathbb{H}$ such that*

$$D(A) = D(A^*) \quad \textit{and} \quad D(B) = D(B^*)$$

One supposes that $\overline{D(A) \cap D(B)} = \mathbb{H}$ and that the closure $\overline{A+B}$ of $A+B$ is a maximal operator. Then we have

$$D((\overline{A+B})^{\frac{1}{2}}) = D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}}) = D((\overline{A+B})^{*\frac{1}{2}})$$

Proof. Let Φ and Ψ be the densely defined closed sectorial sesquilinear forms associated with A and B respectively. Consider their sum $\Xi = \Phi + \Psi$; since $D(A+B) \subset D(\Phi) \cap D(\Psi)$ and that $D(A) \cap D(B)$ is dense in H. It turns out that Ξ is a densely defined closed sectorial sesquilinear form on $\mathbb{H}$. Now, $\overline{A+B}$ is a maximal operator by assumption; it follows that $\overline{A+B}$ is the operator associated with the sesquilinear form Ξ. In the same way, $(\overline{A+B})^*$ is the operator associated with the conjugate Ξ^* of Ξ.

Now, $D(A) \cap D(B) = D(A^*) \cap D(B^*)$ with equivalent norms. From the general fact that $A^* + B^* \subset (A+B)^*$. It follows that $D(\overline{A+B}) \subseteq D((\overline{A+B})^*)$. Thus,

$$D((\overline{A+B})^{\frac{1}{2}}) \subseteq D((\overline{A+B})^{*\frac{1}{2}})$$

Using [18, Theorem 5.2, p. 238], we obtain that

$$D((\overline{A+B})^{\frac{1}{2}}) \subseteq D(\Xi) \subseteq D((\overline{A+B})^{*\frac{1}{2}}) \tag{9}$$

Since $\overline{A+B}$ is m-accretive. Then, substituting $\overline{A+B}$ by $(\overline{A+B})^*$ in (9) yields

$$D((\overline{A+B})^{*\frac{1}{2}}) \subseteq D(\Xi^*) \subseteq D((\overline{A+B})^{\frac{1}{2}}) \tag{10}$$

Comparing (9) and (10), and using the fact that $D(\Xi) = D(\Xi^*)$. It follows that, $D((\overline{A+B})^{\frac{1}{2}}) = D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}}) = D((\overline{A+B})^{*\frac{1}{2}})$. □

Remark 3.2. Since A and B are respectively m-sectorial; then there $0 \leq \alpha, \alpha' < \dfrac{\pi}{2}$ such that $\Theta(A) \subset \mathcal{S}_{\alpha,0}$ and $\Theta(B) \subset \mathcal{S}_{\alpha',0}$. Setting $\beta = \tan\alpha$ and $\beta' = \tan\alpha'$; then:

$$|\Im m\ \Xi(u,u)| \leq \max(\beta, \beta')\ \Re e\ \Xi(u,u), \quad \forall u \in D(\Xi) = D(\Phi) \cap D(\Psi)$$

As consequence, we shall apply theorem 3.1 to the case of unbounded normal operators.

Let A and B be unbounded normal operators on $\mathbb{H}$. According to the spectral theory for unbounded normal operator, one can write

$$A = A_1 - iA_2 \ \text{ and } \ B = B_1 - iB_2,$$

with A_k, B_k self-adjoint operators on $\mathbb{H}$ ($k = 1, 2$), see, e.g., [24, pp. 348-355]. Now since $D(A) = D(A^*)$ and $D(B) = D(B^*)$, it turns out that

$$A^* = A_1 + iA_2 \text{ and } B^* = B_1 + iB_2$$

Also, if one supposes that A_k, B_k to be nonnegative self-adjoint operators ($k = 1, 2$). Then (iA) and (iB) are respectively seen as m-accretive operators, see, e.g., [23, Corollary 4.4, p. 15]. Now let us make the following assumptions

(i) $\overline{D(A) \cap D(B)} = \mathbb{H}$

(ii) $\exists\, C > 0 : \ \langle A_2 u, u\rangle \ \leq \ C\,\langle A_1 u, u\rangle, \ \forall u \in D(A_1^{\frac{1}{2}}) \cap D(A_2^{\frac{1}{2}})$

(iii) $\exists\, C' > 0 : \ \langle B_2 u, u\rangle \ \leq \ C'\,\langle B_1 u, u\rangle, \ \forall u \in D(B_1^{\frac{1}{2}}) \cap D(B_2^{\frac{1}{2}})$

Here, we set $\Lambda = D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}})$.

Corollary 3.3. Let $A = A_1 - iA_2$ and $B = B_1 - iB_2$ be unbounded normal operators on $\mathbb{H}$ such that A_k and B_k are nonnegative ($k = 1, 2$). Assume that assumptions (i), (ii), and (iii) hold and that $\overline{A + B}$ is maximal. Then

$$D(\overline{A+B}^{\frac{1}{2}}) = D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}}) = D(\overline{A+B}^{*\frac{1}{2}})$$

Proof. Let Ξ the sesquilinear form defined by

$$\Xi(u, v) = \langle (A + B)u, v\rangle, \ \ \forall\, u \in D(A) \cap D(B), \ \ v \in \Lambda$$

Consider the Pre-Hilbert space $\mathbb{H}_\Xi = (\Lambda\ ,\ <,>_\Xi)$, where

$$\langle u, v\rangle_\Xi := \langle u, v\rangle_{\mathbb{H}} + \Re e\,\Xi(u, v), \ \ \forall u, v \in \Lambda$$

Since the sum form operator $A_1 \dot{+} B_1$ is a nonnegative self-adjoint operator. It easily follows that $\mathbb{H}_\Xi$ is a Hilbert space. Thus, the sesquilinear form Ξ is closed. Moreover, $D(\Xi) = \Lambda$ is dense in $\mathbb{H}$ ($D(A) \cap D(B) \subset \Lambda$ and (i) holds). From the assumptions (ii) and (iii), we conclude that Ξ is sectorial. Thus, Ξ is a densely defined closed sectorial sesquilinear form. According to theorem 3.1, we know that $\overline{A + B}$ is the m-sectorial operator associated with Ξ. Since $D(A) = D(A^*)$ and $D(B) = D(B^*)$, we complete the proof, using similar arguments as in the proof of the theorem 3.1. □

Let Φ and Ψ be densely defined closed sectorial sesquilinear forms on $\mathbb{H}$. Assume that A and B are respectively the m-sectorial operators associated with Φ and Ψ by the first representation theorem (see [16, Theorem 2.1, p. 322]). Setting $\Xi = \Phi + \Psi$, then we have

Theorem 3.4. *Under previous assumptions. One supposes that A, B satisfy (1) and that $\overline{D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}})} = \mathbb{H}$. In addition if $D(\Xi)$ is closed in the interpolation space $[\mathbb{H}_\Xi, \mathbb{H}]_{\frac{1}{2}}$. Then there exists a unique m-sectorial operator $A \dotplus B$ such that*

$$D((A \dotplus B)^{\frac{1}{2}}) = D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}}) = D((A \dotplus B)^{*\frac{1}{2}})$$

Proof. Since $D(\Xi) = D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}})$ is dense in $\mathbb{H}$. It easily follows that Ξ is a densely defined closed sectorial form. According to Kato's first representation theorem (see [16, Theorem 2.1. p. 322]): there exists a unique m-sectorial operator $A \dotplus B$ associated with Ξ and that $D(A \dotplus B) \subset D(\Xi) = \mathbb{H}_\Xi$, $D((A \dotplus B)^*) \subset D(\Xi) = \mathbb{H}_\Xi$. Since $\mathbb{H}_\Xi \hookrightarrow \mathbb{H}$ is continuous and that $D(\Xi)$ is closed in $[\mathbb{H}_\Xi, \mathbb{H}]_{\frac{1}{2}}$. We complete the proof using the theorem of Lions (theorem 2.7). ∎

Theorem 3.5. *Let $\alpha \in [0, 1)$ and let A and B be invertible m-accretive linear operators on $\mathbb{H}$ such that $\overline{D(A) \cap D(B)} = \mathbb{H}$. One supposes that $\Theta(A) \subseteq S_{\alpha,\beta}$ and $\Theta(B) \subseteq S_{\alpha,\beta'}$, where $\alpha \in [0, \frac{\pi}{2})$ and $\beta, \beta' > 0$. In addition, assume that $\overline{A + B}$ is m-accretive. Then*

(i) $D((\overline{A + B})^{\frac{1}{2}}) = D((\overline{A + B})^{*\frac{1}{2}})$,

(ii) $\Theta(\overline{A + B}) \subseteq S_{\alpha,2\max(\beta,\beta')}$.

Proof. By assumption $\Theta(A) \subseteq S_{\alpha,\beta}$ and $\Theta(B) \subseteq S_{\alpha,\beta'}$. Thus, we have

$$|\Im m < Au, u > | \leq \beta[\, \Re e < Au, u >]^\alpha, \quad \forall u \in D(A) \tag{11}$$

$$|\Im m < Bu, u > | \leq \beta'[\, \Re e < Bu, u >]^\alpha, \quad \forall u \in D(B) \tag{12}$$

It turns out that, $\forall u \in D(A) \cap D(B)$, there exists $\gamma = \max(\beta, \beta')$ such that,

$$(13) |\Im m < (A+B)u, u > | \leq \gamma[(\Re e < Au, u >)^{\alpha} + (\Re e < Bu, u >)^{\alpha}]$$

Now note that the following holds: let $\mu \in [0,1]$ and let $x, y \ \geq 0$. Then

$$x^{\mu} + y^{\mu} \leq 2^{1-\mu}(x+y)^{\mu} \leq 2(x+y)^{\mu} \tag{14}$$

Applying "(14)" to (13), and by density, we have: $\forall u \in D(\overline{A+B})$

$$|\Im m < \overline{A+B}u, u > | \ \leq 2\gamma \ [\Re e < \overline{A+B}u, u >]^{\alpha} \tag{15}$$

Since $\overline{A+B}$ is m-accretive, we use (15) and [22, Theorem B, p. 257-258] to obtain the sought result, that is:

$$D((\overline{A+B})^{\frac{1}{2}}) = D((\overline{A+B})^{*\frac{1}{2}} \tag{16}$$

From (15), it easily follows that $\Theta(\overline{A+B}) \subseteq S_{\alpha, 2\max(\beta,\beta')}$. □

In what follows, we consider A, B be invertible m-accretive operators on $\mathbb{H}$ satisfying

$$\Theta(A) \subseteq S_{\alpha,\beta} \ \text{ and } \ \Theta(B) \subseteq S_{\alpha,\beta'}, \tag{17}$$

where $\alpha \in [0,1)$ and $\beta, \beta' > 0$. Let Φ and Ψ be the sesquilinear forms associated with A and B, respectively. From (17), it follows that A and B verify (1). Thus, Φ and Ψ can be decomposed as

$$\Phi(u,v) = \langle A^{\frac{1}{2}}u, A^{*\frac{1}{2}}v \rangle, \quad u, v \in D(A^{\frac{1}{2}}) = D(\Phi) = D(A^{*\frac{1}{2}}), \tag{18}$$

$$\Psi(u,v) = \langle B^{\frac{1}{2}}u, B^{*\frac{1}{2}}v \rangle \quad u, v \in D(B^{\frac{1}{2}}) = D(\Psi) = D(B^{*\frac{1}{2}}). \tag{19}$$

Now consider their sum, $\Xi = \Phi + \Psi$. Thus, $\forall \ u, v \in D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}})$,

$$\Xi(u,v) =< A^{\frac{1}{2}}u, A^{*\frac{1}{2}}v > + < B^{\frac{1}{2}}u, B^{*\frac{1}{2}}v > \tag{20}$$

It is not hard to see that $\Theta(\Xi) \subset S_{\alpha,\gamma}$, where $\alpha \in [0,1)$ is given above and $\gamma = 2\max(\beta,\beta') > 0$. Now, let $A \dot{+} B$ be the operator associated with Ξ. Thus, we formulate this fact as follows.

Theorem 3.6. *Under previous assumptions; assume* $\overline{D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}})} = \mathbb{H}$ *and that the operator* $A \dot{+} B$ *defined above is (invertible) m-accretive. Then*

$$D((A \dot{+} B)^{\frac{1}{2}}) = D((A \dot{+} B)^{*\frac{1}{2}})$$

Proof. .- Since $A \dot{+} B$ is an invertible m-accretive operator satisfying $\sigma(A \dot{+} B) \subset \Theta(A \dot{+} B) \subset S_{\alpha,\gamma}$, where $\alpha \in [0, 1)$ and $\gamma = 2\max(\beta, \beta') > 0$. One completes the proof using a result due to McIntosh [22, Theorem B, p. 257-258].

□

4. Applications

This section is concerned with the perturbed Schrödinger operators. Indeed, we shall show that the perturbed operator $S_Z = -Z\Delta + V$ verifies the square root problem of Kato, under suitable assumptions on the complex number Z and the singular complex potential V. The operator S_Z will be seen as the algebraic sum of two m-sectorial operators A_Z and B that we will define in the sequel with the help of sesquilinear forms.

Let $\Omega \subset \mathbb{R}^d$ be an open subset and set $\mathbb{H} = L^2(\Omega)$. Let Φ_Z be the sesquilinear form defined by

$$\Phi_Z(u, v) = \int_\Omega Z\nabla u\overline{\nabla v}\, dx, \quad \forall\, u, v \in D(\Phi_Z) = \mathbb{H}^1_0(\Omega), \tag{21}$$

where $Z = \alpha - i\beta$ ($\alpha, \beta \in \mathbb{R}$) is a complex number satisfying

$$\alpha, \beta > 0 \quad \text{and} \quad \beta \leq \alpha \tag{22}$$

Clearly, the assumption (22) implies that Φ_Z is a sectorial sesquilinear form on $L^2(\Omega)$.

Let V be a measurable complex-valued function and let Ψ be the sesquilinear form given by

$$\Psi(u, v) = \int_\Omega V u\overline{v} dx, \quad \forall\, u, v \in D(\Psi), \tag{23}$$

where $D(\Psi) = \{u \in L^2(\Omega): \; V|u|^2 \in L^1(\Omega)\}$. Throughout this section we assume that the potential $V \in L^1_{loc}(\Omega)$ and that there exists $\theta \in (0, \frac{\pi}{2})$ such that

$$|\arg(V(x))| \leq \theta, \quad \text{almost everywhere} \tag{24}$$

From (24), it turns out that

$$|\Im m \; \Psi(u,u)| \leq \; \tan\theta \, \Re e \; \Psi(u,u), \quad \forall \, u \in D(\Psi) \tag{25}$$

In other words, Ψ is a sectorial sesquilinear from on $L^2(\Omega)$.

Under the previous assumptions, Φ and Ψ are respectively, densely defined closed sectorial forms. The operators associated with both Φ_Z and Ψ are respectively given by

$$D(A_Z) = \{u \in \mathbb{H}^1_0(\Omega) : Z\Delta u \in L^2(\Omega)\}, \quad A_Z u = -Z\Delta u, \;\; \forall \, u \in D(A_Z)$$

$$D(B) = \{u \in L^2(\Omega) : \; Vu \in L^2(\Omega)\}, \;\; Bu = Vu, \;\; \forall \, u \in D(B)$$

It is not hard to see that A_Z and B are respectively unbounded normal operators on $L^2(\Omega)$ and that they can be expressed as: $A_Z = A^1_Z - iA^2_Z$, where $A^1_Z = -\alpha\Delta$ and $A^2_Z = -\beta\Delta$ are nonnegative self-adjoint operators, and $B = B^1_V - iB^2_V$, where B^1_V, B^2_V are nonnegative self-adjoint operators.

Assume that $\Omega = \mathbb{R}^d$. It will be seen that $\overline{D(A_Z) \cap D(B)} = L^2(\mathbb{R}^d)$. Consider the sum $\Xi_Z = \Phi_Z + \Psi$. Clearly, Ξ_Z is a densely defined closed sectorial form. Since $\overline{-Z\Delta + V}$ is m-sectorial (see [4]). It follows that $\overline{-Z\Delta + V}$ is the operator associated with Ξ_Z. In fact, Brézis and Kato computed it in [4]. It is defined by

$$D(\overline{-Z\Delta + V}) = \{u \in \mathbb{H}^1(\mathbb{R}^d) : \; V|u|^2 \in L^1(\mathbb{R}^d) \text{ and } -Z\Delta u + Vu \in L^2(\mathbb{R}^d)\}$$

$$\overline{-Z\Delta + V}u = -Z\Delta u + Vu, \; \forall u \in D(\overline{-Z\Delta + V})$$

Let us notice that $D(A_Z) = \mathbb{H}^2(\mathbb{R}^d)$ and $D(B) = \{u \in L^2(\mathbb{R}^d) : \;\; Vu \in L^2(\mathbb{R}^d)\}$, and their intersection is dense in $L^2(\mathbb{R}^d)$. Therefore applying Corollary 3.3 to A_Z and B. It easily follows that

$$D((\overline{-Z\Delta + V})^{\frac{1}{2}}) = \mathbb{H}^1(\mathbb{R}^d) \cap D(B^{\frac{1}{2}}) = D((\overline{-Z\Delta + V})^{*\frac{1}{2}}) \tag{26}$$

In particular where $d = 1$. Then we obtain that

$$D((\overline{-Z\Delta + V})^{\frac{1}{2}}) = \mathbb{H}^1(\mathbb{R}) = D((\overline{-Z\Delta + V})^{*\frac{1}{2}}) \tag{27}$$

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